8) General Coupled mode theory

Introduction

* Closed resonators, 1 dof

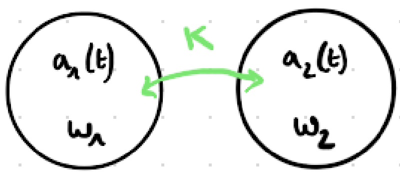
$a(t)$ complex amplitude of the field, ω_0 resonance frequency

$|a|^2$ is the energy in the field

$$a(t) = A e^{j\omega_0 t} \Rightarrow \mathring{a}(t) = j\omega_0 a(t) \Rightarrow -j\mathring{a}(t) = \omega_0 a(t)$$

damping γ? $\Rightarrow -j\,\mathring{a}(t) = (\omega_0 + j\gamma)\, a(t)$ → simple toy model

* 2 d.o.f.



$$\vec{a}(t) = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} \qquad -j\,\mathring{\vec{a}} = \Omega \vec{a}, \quad \Omega = \begin{pmatrix} \omega_1 & \cancel{0}\ K \\ \cancel{0}\ K & \omega_2 \end{pmatrix}$$

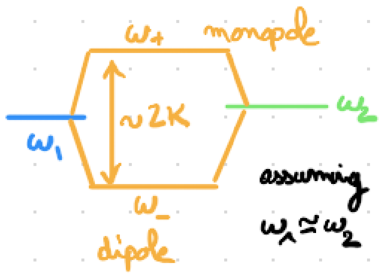
analog to $-j\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle$, Schrödinger, $\hat{H} \leftrightarrow \Omega$, $|\psi\rangle \leftrightarrow \vec{a}$

lossless $\Rightarrow$ Ω Hermitian $\Rightarrow K \in \mathbb{R}$

eigenmodes $\Rightarrow$ $\vec{a}(t) = \vec{a} e^{j\omega t}$ (self-sustained oscillations)

$$\Rightarrow \Omega \vec{a} = -j(j\omega\vec{a}) \Rightarrow \Omega\vec{a} = \omega\vec{a} \Rightarrow \begin{cases} \omega \text{ is eigenvalue of } \Omega \\ \vec{a} \text{ is eigenvector} \end{cases}$$

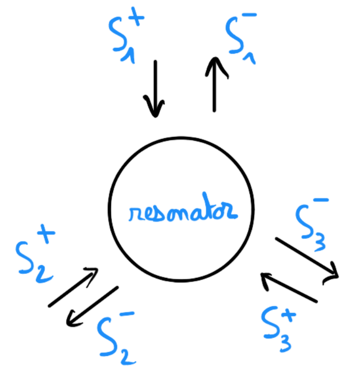
$$\omega_\pm = \frac{\omega_1 + \omega_2}{2} \pm \sqrt{\frac{(\omega_2 - \omega_1)^2}{4} + K^2}$$



What happens when we probe such resonant systems by scattering?

Link between Ω and S? $\Rightarrow$ coupled-mode theory

1. Model of the closed resonator



$e^{j\omega t}$ time convention
n cavity modes, possibly coupled
amplitudes a_i, $|a_i|^2$ mode energy
m ports
$\vec{a} = [a_1, a_2, \dots, a_n]^T$

Closed resonator model

$$\frac{d\vec{a}}{dt} = j\Omega\vec{a}$$ when lossless, with Ω Hermitian

proof: $$\frac{dE_{tot}}{dt} = 0 = \frac{d\vec{a}^+\vec{a}}{dt} = \frac{d\vec{a}^+}{dt}\vec{a} + \vec{a}^+\frac{d\vec{a}}{dt} = -j\vec{a}^+\Omega\vec{a} + j\vec{a}^+\Omega\vec{a}$$

$$= -j\vec{a}^+(\Omega - \Omega^+)\vec{a} \Rightarrow \underline{\Omega = \Omega^+}$$ QED

Lossy closed resonator model (intrinsic losses)

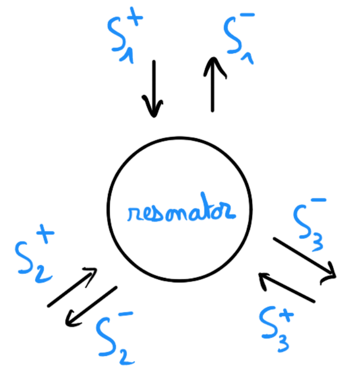
$$\frac{d\vec{a}}{dt} = (j\Omega - \Gamma_L)\vec{a}$$ with Γ_L Hermitian

$\Rightarrow$ Γ_L must have only positive eigenvalues

proof: $\vec{a}(t) = e^{(j\Omega - \Gamma_L)t}\vec{a}(0) = e^{j\Omega t} e^{-\Gamma_L t}\vec{a}(0)$ must be stable

$e^{j\Omega t}$ is unitary since Ω is Hermitian $\rightarrow$ stable

$e^{-\Gamma_L t}$ is stable $\Leftrightarrow$ Γ_L has positive eigenvalues only QED



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$\vec{a} = [a_1, a_2, \dots, a_n]^T$

Closed resonator model

$$\frac{d\vec{a}}{dt} = j\Omega\vec{a} \quad \text{when lossless, with } \Omega \text{ Hermitian}$$

proof: $$\frac{dE_{tot}}{dt} = 0 = \frac{d\vec{a}^+\vec{a}}{dt} = \frac{d\vec{a}^+}{dt}\vec{a} + \vec{a}^+\frac{d\vec{a}}{dt} = -j\vec{a}^+\Omega\vec{a} + j\vec{a}^+\Omega\vec{a}$$

$$= -j\vec{a}^+(\Omega - \Omega^+)\vec{a} \Rightarrow \underline{\Omega = \Omega^+} \quad \text{QED}$$

Lossy closed resonator model (intrinsic losses)

$$\frac{d\vec{a}}{dt} = (j\Omega - \Gamma_L)\vec{a} \quad \text{with } \Gamma_L \text{ Hermitian}$$

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proof: $\vec{a}(t) = e^{(j\Omega - \Gamma_L)t}\vec{a}(0) = e^{j\Omega t}e^{-\Gamma_L t}\vec{a}(0)$ must be stable

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2. Coupling to m ports

$$\begin{cases} \dfrac{d\vec{a}}{dt} = (j\Omega - \Gamma_L - \Gamma_R)\vec{a} + K^T \vec{S}_+ & (1) \\ \vec{S}_- = C\vec{S}_+ + D\vec{a} & (2) \end{cases}$$

Γ_R : $m \times m$ Hermitian matrix describing radiative decay to the ports

$\vec{S}_+$: $m \times 1$ vector of input signal amplitudes, normalized such that $S^*_{j+} S_{j+}$ is the power incident at port j.

$\vec{S}_-$: $m \times 1$ " output ". $S^*_{j-} S_j$ outgoing power at port j

K^T : $m \times m$ Coupling matrix from $\vec{S}_+$ to $\vec{a}$

D : $m \times m$ Coupling matrix from $\vec{a}$ to $\vec{S}_-$

C : $m \times m$ Unitary matrix describing the scattering when the resonator is not excited (direct scattering path)

3. General Constraints

$$\boxed{\begin{aligned} &D^\dagger D = 2\Gamma_R && (C0)\\ &\tilde{C} = C^T && (C1)\\ &\tilde{\Gamma}_R = \Gamma_R^* && (C2)\\ &\tilde{D} = K && (C3)\\ &\tilde{K} = D && (C4)\\ &C^T D^* = -K && (C5)\\ &D^\dagger D = K^\dagger K && (C6)\end{aligned}}$$

where $\sim$ denotes the time-reversed image

Assumptions:
- C is unitary
- energy conservation
- LTI systems with linewidth $\ll$ resonance freq

If time reversal symmetry is preserved, $\tilde{C} = C$, $\tilde{D} = D$, $\tilde{K} = K$, $\tilde{\Gamma}_R = \Gamma_R$ and the system becomes

$$\boxed{\begin{aligned} &D^\dagger D = 2\Gamma_R\\ &K = D\\ &C D^* = -D\end{aligned}}$$

as previously shown in IJQE 40, 10, 2004.

3. Harmonic oscillations

Assume $\vec{a} = e^{j\omega t}\vec{A}$ with ω a $m \times m$ complex matrix

$$\Rightarrow \frac{da}{dt} = j\omega\vec{a} = (j\Omega - \Gamma_L - \Gamma_R)\vec{a} + K^T\vec{S}^+$$

$$\Rightarrow \boxed{(j(\omega - \Omega) + \Gamma)\vec{a} = K^T\vec{S}^+} \quad (3)$$

with $\underline{\Gamma = \Gamma_R + \Gamma_L}$

4. Scattering matrix

Assume $\vec{S}^+ = e^{j\omega t} \vec{s}^+$ with $\omega \in \mathbb{R}$ (or $\mathbb{C}$)

then by linearity $\vec{a} = e^{j\omega t} \vec{A}$ $\quad \omega = \omega \mathbb{1}_m$

$\Rightarrow \left(j(\omega \mathbb{1}_m - \Omega) + \Gamma \right) \vec{a} = K^T \vec{S}_+$

Assuming $\left(j(\omega \mathbb{1}_m - \Omega) + \Gamma \right)^{-1}$ exists :

$$\boxed{\vec{a} = \left(j(\omega \mathbb{1}_m - \Omega) + \Gamma \right)^{-1} K^T \vec{S}_+} \quad (4)$$

Then

$$\vec{S}_- = S\,\vec{S}_+ = C + D\left[j(\omega \mathbb{1}_m - \Omega) + \Gamma \right]^{-1} K^T \vec{S}_+$$

$$\Rightarrow \quad \boxed{S = C + D\left[j(\omega \mathbb{1}_m - \Omega) + \Gamma \right]^{-1} K^T} \quad (5)$$

5. Bound States in Continuum

If $j(\omega \mathbb{1}_m - \Omega) + \Gamma_R + \Gamma_L$ is not invertible then it is not possible to determine $\vec{a}$ from $\vec{S}_+ \Rightarrow \vec{a}$ is decoupled from $\vec{S}_+$

Actually, since $j(\omega - \Omega) + \Gamma_R + \Gamma_L$ is not invertible, its determinant is zero $\Rightarrow \exists$ a zero eigenvalue $\Rightarrow \exists\, \vec{a}_\infty$ /

$$j\,\left(j(\omega \mathbb{1}_m - \Omega) + \Gamma_R + \Gamma_L \right) \vec{a}_\infty = \vec{0}$$

$$\Rightarrow \Big[\underbrace{\Omega + j(\Gamma_R + \Gamma_L)}_{\text{non Hermitian Hamiltonian}} \Big] \vec{a}_\infty = \underbrace{\omega}_{\in \mathbb{R}}\, \vec{a}_\infty$$

$\Rightarrow \vec{a}_\infty$ has infinite lifetime $\Rightarrow$ BiC

Rq: $\vec{a}$ can be any linear superposition of $\vec{a}^{\,i}_\infty$, if $\exists$ several.

6. Time-bandwidth limit

When exciting at ω, the stored energy is aa^{+} Using (4):

$$a^{+}a = (K^T\vec{S}_+)^{+}\left[-j(\omega \mathbb{1}_m - \Omega^{+}) + \Gamma^{+}\right]^{-1}\left[j(\omega \mathbb{1}_m - \Omega) + \Gamma\right]^{-1} K^T\vec{S}_+$$

$$\Rightarrow a^{+}a = (K^T\vec{S}_+)^{+}\left[\left(j(\omega \mathbb{1}_m - \Omega) + \Gamma\right)\left(-j(\omega \mathbb{1}_m - \cancel{\Omega^{+}}) + \cancel{\Gamma^{+}}\right)\right]^{-1} K^T\vec{S}_+$$

hermitian

$$\Rightarrow a^{+}a = (K^T\vec{S}_+)^{+}\left[(\omega \mathbb{1}_m - \Omega)^2 + \Gamma^2\right]^{-1} K^T\vec{S}_+$$

If only one resonator is present $(m=1)$: $\Omega = \omega_0 > 0$, $\Gamma = \gamma > 0$

$$\Rightarrow a^{+}a = \frac{|K^T\vec{S}_+|^2}{(\omega-\omega_0)^2 + \gamma^2}$$

In f domain: Lorentzian with FWHM $= 2\gamma = \Delta\omega$ bandwidth

In t domain: Lifetime $1/\gamma = \Delta t$ decay time

Time bandwidth product: $\boxed{\Delta\omega \Delta t = 2}$ $\forall$ linear system obeying the eqt of motion (4)

Quality factor: $Q = \frac{\omega_0 \Delta t}{2} = \frac{\omega_0}{\Delta\omega}$ Quality factor

6. Proof of C1

For C alone (no resonators) :

$$\vec{S}_- = C\,\vec{S}_+$$

By time-reversal $\quad \vec{S}_+^{\,*} = \tilde{C}\,\vec{S}_-^{\,*} \quad \Rightarrow \quad \vec{S}_+ = \tilde{C}^*\,\vec{S}_-$

$$\Rightarrow \quad (\tilde{C}^*)^{-1}\,\vec{S}_+ = \vec{S}_-$$

$$\Rightarrow \qquad C = (\tilde{C}^*)^{-1}$$

Assuming the direct pathway is lossless C is unitary, and so does $\tilde{C}$

$\Rightarrow \qquad C = \tilde{C}^{*\dagger} = \tilde{C}^T \quad \Rightarrow \quad \tilde{C} = C^T \qquad$ QED

6. Proof of C0: Resonator decay

$$\vec{S}_+^d = 0 \qquad \vec{a}_d(t=0) = \vec{a}_d(0)$$

$$\begin{cases} \dfrac{d\vec{a}_d}{dt} = (j\Omega - \Gamma)\,\vec{a}_d \\ \vec{S}_-^d = D\,\vec{a}_d \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} \vec{a}_d = e^{(j\Omega - \Gamma)t}\,\vec{a}_d(0) \longrightarrow \vec{0} \quad [t \to +\infty] \\ \vec{S}_-^d = D\,\vec{a}_d \end{cases}}$$

<u>Energy decay</u>:

$$\frac{d\vec{a}_d^+\vec{a}_d}{dt} = \frac{d\vec{a}_d^+}{dt}\vec{a}_d + \vec{a}_d^+\frac{d\vec{a}_d}{dt} = \vec{a}_d^+(-j\Omega^+ - \Gamma_R^+ - \Gamma_L^+)\vec{a}_d + \vec{a}_d^+(j\Omega - \Gamma_R - \Gamma_L)\vec{a}_d$$

$$= \underbrace{\vec{a}_d^+(-2\Gamma_R)\vec{a}_d}_{\text{radiation to ports}} + \underbrace{\vec{a}_d^+(-2\Gamma_L)\vec{a}_d}_{\text{other (intrinsic absorption, etc)}}$$

By power conservation $\vec{a}_d^+(-2\Gamma_R)\vec{a}_d^+ = -\vec{S}_-^{d+}\vec{S}_-^d = -\vec{a}_d^+ D^+ D\,\vec{a}_d$

$$\Rightarrow \boxed{D^+D = 2\Gamma_R}$$ QED energy conservation

* generalisation of the notion of decay time: In the basis where Γ is diagonal:

$$\frac{d\vec{a}^+\vec{a}}{dt} = \vec{a}^+(-2\Gamma)\vec{a} = \vec{a}^+\begin{pmatrix} -2\gamma_1 & & 0 \\ & -2\gamma_2 & \\ 0 & & \ddots\; -2\gamma_n \end{pmatrix}\vec{a} = -2\,\mathrm{Tr}\,\Gamma\,\vec{a}^+\vec{a}$$

$$\Rightarrow \boxed{\Delta t = \frac{1}{\mathrm{Tr}\,\Gamma}}$$

7. Proof of C2-4: Time-reversal of decay

$$T: \quad \vec{S}_+ \longrightarrow \vec{S}_-^*$$
$$\vec{S}_- \longrightarrow \vec{S}_+^*$$
$$\vec{a} \longrightarrow \vec{a}^*$$
$$\Omega, C, K, D, r \longrightarrow \tilde{\Omega}, \tilde{C}, \tilde{K}, \tilde{D}, \tilde{r}$$

The time reversed decay process is an exponential growth with

$$\begin{cases} \vec{S}_+^g = \vec{S}_-^{d*} = (D\vec{a}_d)^* \\ \vec{a}_g = \vec{a}_d^* \\ \vec{S}_-^g = 0 \end{cases}$$

Consider eq (2) during this growth process

$$0 = \tilde{C}\vec{S}_+^g + \tilde{D}\vec{a}_g = \tilde{C}\vec{S}_-^{d*} + \tilde{D}\vec{a}_d^* = \tilde{C}D^*\vec{a}_d^* + \tilde{D}\vec{a}_d^*$$

$$\Rightarrow \boxed{\tilde{C}D^* = -\tilde{D}} \quad (7)$$

Since $D^\dagger D \overset{(6)}{=} 2r_R$, $\tilde{D}^\dagger\tilde{D} = 2\tilde{r}_R \overset{(7)}{=} (-D^{*\dagger}\tilde{C}^\dagger)(-\tilde{C}D^*)$

$$= D^{*\dagger}\tilde{C}^\dagger\tilde{C}D^* = (D^\dagger D)^* = 2r_R^*$$

We have used: C is unitary: $C^\dagger C = 1 = \tilde{C}^\dagger\tilde{C}$

$$\Rightarrow \boxed{\tilde{r}_R = r_R^*}$$

QED

Consider now the time-reversed decay of a lossless resonator ($r_L = 0$)

During the time reversed decay, the amplitude $\vec{a}_g$ follows a time evolution with $\omega = \tilde{\Omega} - j\tilde{r}_R$

We can use eq (3) with $\vec{a}_g = \vec{a}_d^{\,*}$, $\omega = \tilde{\Omega} - j\tilde{r}_R$, $\vec{S}_1^{+} = D^*\vec{a}_d^{\,*}$

$$\Rightarrow \left[j(\omega - \tilde{\Omega}) + \tilde{r}_R\right] \vec{a}_d^{\,*} = \tilde{K}^T D^* \vec{a}_d^{\,*}$$

$$\Rightarrow \left[j(-j\tilde{r}_R) + \tilde{r}_R\right] \vec{a}_d^{\,*} = \tilde{K}^T D^* \vec{a}_d^{\,*}$$

$$\Rightarrow \quad \tilde{r}_R + \tilde{r}_R = \tilde{K}^T D^* = 2 r_R^*$$

$$\overset{*}{\Rightarrow} \quad \tilde{K}^+ D = 2 r_R = D^T D \quad \Rightarrow \quad (\tilde{K}^+ - D^+) D = 0$$

$$\Rightarrow \quad \boxed{\tilde{K} = D} \quad \Rightarrow \quad \tilde{D} = K \quad \text{QED}$$

5. Proof of C5: obvious from (7) and (C4)
6. Proof of C6

$$(C5) \Rightarrow \quad C^T D^* = -K \quad \Rightarrow \quad C^+ D = -K^* \Rightarrow D = -CK^*$$

$$\Rightarrow \begin{cases} D = -CK^* \\ D^+ = -K^T C^+ \end{cases}$$

$$\Rightarrow D^+ D = +K^T C^+ C K^* = K^T K^* = (\underbrace{K^+ K}_{>0})^* = K^+ K \quad \text{QED}$$

* Rq: This is a fluctuation dissipation relation.